Undecidability of local structures of $s$-degrees and $Q$-degrees

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Abstract

We show that the first order theory of the $\Sigma^0_2$ $s$-degrees is undecidable. Via isomorphism of the $s$-degrees with the $Q$-degrees, this also shows that the first order theory of the $\Pi^0_2 Q$-degrees is undecidable. Together with a result of Nies, the proof of the undecidability of the $\Sigma^0_2 s$-degrees yields a new proof of the known fact (due to Downey, LaForte and Nies) that the first order theory of the c.e. $Q$-degrees is undecidable.

1 Introduction

Cooper, [6], asks to characterize the degree of the first order theory of the $\Sigma^0_2$ $s$-degrees. We are not able to fully answer this question, but we are able to show that this theory is undecidable. Undecidability follows from the following two facts, which hold in the $\Sigma^0_2 s$-degrees: there is an independent antichain which is first order definable with three parameters (Theorem 2.2); and a suitable version of the Exact Degree Theorem of Nies (Theorem 2.1). In addition, Theorem 2.2 together with another suitable version of the Nies Exact Degree Theorem yields the undecidability of the $\Pi^0_2 s$-degrees. Via isomorphism of the $s$-degrees with the $Q$-degrees, this also gives undecidability of the structure of the $\Pi^0_2 Q$-degrees, and (a result of Downey, LaForte and Nies [19]) undecidability of the c.e. $Q$-degrees.

Positive reducibilities formalize models of relative computability which use only “positive” oracle information. The most comprehensive positive reducibility is enumeration reducibility, denoted by $\leq e$. Intuitively a set $A$ is enumeration reducibility, denoted by $\leq e$. Intuitively a set $A$ is

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reducible to a set $B$ if there is some effective procedure for enumerating $A$ given any enumeration of $B$. Following Friedberg and Rogers [8], this is made mathematically precise by defining $A \leq_e B$ if there exists a c.e. set $\Phi$ such that

$$A = \{x : (\exists \text{ finite } F)(\langle x, F \rangle \in W \& F \subseteq B)\}$$

(often denoted by $A = \Phi^B$) where finite sets are identified with their canonical indices. In this context a c.e. set $\Phi$ is also called an *enumeration operator*. According to this definition, a computation may enumerate a number $x$ in $A$ only upon retrieval of positive information about $B$, i.e. information of the form $F \subseteq B$, for some pseudo–pair $\langle x, F \rangle \in \Phi$. Access to positive information about $B$ is made possible via some enumeration of $B$.

### 1.1 $s$-reducibility

It is clear that given a set $B$, an enumeration operator $\Phi$, and a given $x$, there is no bound to the number $n$ of oracle questions which are needed to enumerate $x$ in $\Phi^B$, i.e. to the cardinality of a finite set $F$ for which we need $F \subseteq B$, in order to have $x \in \Phi^B$. One can therefore introduce restricted versions of enumeration reducibility by requesting instead that there be such a bound. Although extreme, the case $n = 1$, in which for any given $x$ we need at most one oracle question, is particularly interesting, and occurs often in practical applications of enumeration reducibility. This suggests the following definition:

**Definition 1.1.** An enumeration operator $\Phi$ is called an *$s$-operator* if for every $\langle x, F \rangle \in \Phi$, we have that $F$ has at most one element.

It is straightforward to see that the $s$-operators ($s$ stands for singleton) can be effectively listed, and give rise to a reducibility (called $s$-reducibility), denoted by $\leq_s$. The corresponding degree structure, denoted by $D_s$, consists of the equivalence classes, called $s$-degrees, of the subsets of $\omega$ under the equivalence relation $\equiv_s$ generated by $\leq_s$. The $s$-degree of a set $A$ will be denoted by $\text{deg}_s(A)$. The structure $D_s$ is an upper semilattice with least element $0_s = \text{deg}_s(\emptyset)$ consisting of the c.e. sets, and the operation of least upper bound is given by $\text{deg}_s(A) \cup \text{deg}_s(B) = \text{deg}_s(A \oplus B)$, where $\oplus$ denotes the usual disjoint union of sets. The reducibility $\leq_s$ is properly contained in $\leq_e$: As shown by Zacharov, [23], every nonzero $e$-degree contains at least two $s$-degrees. The reader is referred to the papers [6], [5], [18] for a survey of results on $s$-reducibility.

### 1.2 $Q$-reducibility

An apparently different but intimately related reducibility is $Q$-reducibility (due to Tennenbaum, as quoted by Rogers [20, p. 159]): A set $A$ is *quasi-reducible* ($Q$-reducible) to a set $B$, $A \leq_Q B$, if and only if there exists a total function $f$ such that $A = \{x : W_{f(x)} \subseteq B\}$. (When dealing with $\leq_Q$, the set $\omega$ should not be considered as lying in the universe of the reducibility, as $\omega <_Q A$, for every set $A \neq \omega$.) In the usual way, the reducibility $\leq_Q$ gives rise to a degree structure,
denoted by \( D_Q \); the elements of \( D_Q \) are called \( Q \)-degrees; the \( Q \)-degree of a set \( A \neq \omega \) will be denoted by \( \deg_Q(A) \). The structure \( D_Q \) is an upper semilattice with least element \( 0_Q = \deg_Q(\emptyset) \) consists of the \( \Pi^0_1 \) sets, and the usual operation of least upper bound.

Several interesting applications of \( Q \)-reducibility to algebra are known. One can for instance quote Dobritsa’s theorem (see [3]) stating that for every set \( X \) there is a word problem having the same \( Q \)-degree of \( X \). Belegradek, [3], shows that a necessary condition for computably presented groups \( G \) and \( H \) to have \( G \) a subgroup of every algebraically closed group of which \( H \) is a subgroup, is that the word problem for \( G \) be \( Q \)-reducible to the word problem of \( H \). It is worth noticing that this condition is also sufficient, [11], if \( \leq_Q \) is replaced by \( \leq_T \). But then since on c.e. sets Turing reducibility implies \( Q \)-reducibility, for computably presented groups with c.e. word problems the same condition is both necessary and sufficient. \( Q \)-reducibility has also been studied in connection with abstract complexity theoretic questions: Blum and Marques in [4] introduced the notions of subcreative and effectively speedable sets and they proved that a recursively enumerable set is subcreative if and only if it is effectively speedable. Gill and Morris in [10] gave a simple and interesting characterization of effectively speedable sets in terms of \( Q \)-complete sets. They proved that an c.e. set is effectively speedable if and only if it is \( Q \)-complete.

There is an extensive bibliography on \( Q \)-reducibility: For c.e. \( Q \)-degrees see for instance [7], [19] and [17]; Arslanov and Omanadze in [2] study the \( Q \)-degrees of \( n \)-c.e. sets.

1.3 \( s \)-reducibility or \( Q \)-reducibility?

The following is a useful result due to Gill and Morris [10] relating \( s \)-reducibility to \( Q \)-reducibility, where given a set \( X \subseteq \omega \), \( \overline{X} \) denotes its complement.

**Lemma 1.2** (Isomorphism Lemma). For any sets \( A \) and \( B \neq \omega \), \( A \leq_Q B \) if and only if \( \overline{A} \leq_s \overline{B} \).

**Proof.** If \( A \) and \( B \) are given, with \( B \neq \omega \), and \( A \leq_Q B \) via a computable function \( f \), then Define

\[
\Gamma = \{ \langle x, \{ y \} \rangle : y \in W_{f(x)} \}.
\]

Then \( \Gamma \) is an \( s \)-operator and \( \overline{A} = \Gamma^{\overline{f}} \).

On the other hand, suppose that \( \overline{A} \leq_s \overline{B} \) via the \( s \)-operator \( \Gamma \). Let \( b \notin B \), and let

\[
W_{f(x)} = \begin{cases} \{ y : \langle x, \{ y \} \rangle \in \Gamma \} & \text{if } \langle x, \emptyset \rangle \notin \Gamma, \\ \{ y : \langle x, \{ y \} \rangle \in \Gamma \} \cup \{ b \} & \text{otherwise}. \end{cases}
\]

Then \( A \leq_Q B \) via the computable function \( f \). \( \square \)

Notwithstanding this isomorphism, \( s \)-reducibility and \( Q \)-reducibility have lived so far quite independent lives. Most of the papers on \( s \)-reducibility do
not mention $Q$-reducibility, and vice versa most of the papers on $Q$-reducibility do not mention $s$-reducibility. An additional bit of confusion comes perhaps from an early unusual variety of approaches to $s$-reducibility: Friedberg and Rogers originally defined $A \leq_s B$ if $A = \{x : W_{f(x)} \cap B \neq \emptyset\}$ for some computable function $f; \leq_s$ appears as $\leq_{se}$ in [12]; the branch finite version of $\leq_s$ (i.e. the reducibility given by $s$-operators $\Phi$ in which for every $x$ there are only finitely many axioms $(x, F) \in \Phi$) appears as $\leq_Q$ in [16]. Our formalization of $s$-reducibility, and the notion of $s$-operator, derives from [10].

Following [18], one can define a jump operation on the $s$-degrees, for which the jump of the least element $\emptyset_s$ is given by the $s$-degree $\emptyset'_s = \text{deg}_s(K)$, where $K$ is the complement of the halting set $K$. In the following, we denote $\mathcal{L}_s = D_s(\leq_s \emptyset'_s)$. The structure $\mathcal{L}_s$ is called the local structure of the $s$-degrees, studied in [22]. It is straightforward to show that for every set $A$, $A \leq_s K$ if and only if $A \in \Sigma^0_2$. Thus the elements of $\mathcal{L}_s$ are exactly the $\Sigma^0_2$ $s$-degrees, and consist only of $\Sigma^0_2$ sets.

Via the isomorphism of Lemma 1.2, this gives also a jump operation on the $Q$-degrees, so that for the first jump $\emptyset'_Q$ we have $\emptyset'_Q = \text{deg}_Q(K)$, and the local structure of the $Q$ degrees consists exactly of the $\Pi^0_2 Q$-degrees.  

Our notations and terminology for computability theory are standard, and can be found in [14], [15], [20], and [21].

2 The theorems

The following result shows how to “code” any $\Sigma^0_4$-set in an independent family of $s$-degrees below $\emptyset'_s = \text{deg}_s(K)$. Recall that in an upper semilattice $(U, \leq, \lor)$, a countable $A \subseteq U$ is called independent if for every $a \in A$ and any finite $F \subset A$, we have

$$a \leq \lor F \Rightarrow a \in F.$$ 

**Theorem 2.1** (Exact Degree Theorem for the $\Sigma^0_2$ $s$-degrees). Suppose that $\{A_i\}_{i \in \omega}$ is a uniformly $\Sigma^0_2$ sequence of sets such that the family $\{\text{deg}_s(A_i)\}_{i \in \omega}$ is independent. Then, for each $\Sigma^0_4$-set $S$, there exists a $\Sigma^0_2$ set $B$ such that

$$i \in S \Leftrightarrow A_i \leq_s B.$$ 

Moreover, the result holds uniformly: a $\Sigma^0_2$ index for $B$ can be uniformly found starting from any $\Sigma^0_4$-index of $S$.

**Proof.** An examination of the proof by Nies in [13] of the Exact Degree Theorem for $\Sigma^0_2 e$-degrees shows that he actually proved that

$$i \in S \Rightarrow A_i \leq_m B$$

$$i \notin S \Rightarrow A_i \not\leq_e B.$$ 

Thus is it straightforward to adapt the proof to the $\Sigma^0_2$ $s$-degrees. \hfill \Box
The Exact Degree Theorem turns out to be quite useful. If we could show that there exists a uniformly $\Sigma^0_2$ sequence of sets whose $s$-degrees form an independent antichain $\{a_i\}_{i \in \omega}$ of $\Sigma^0_2$ s-degrees, which is first order definable with parameters, then this would yield that the first order theory of the $\Sigma^0_2$ s-degrees is undecidable. This is done in the following manner. Assume that $\alpha(v, \overline{p})$ is a first order relation with parameters $\overline{p}$ that defines the elements of an independent antichain. By Theorem 2.1, every $\Sigma^0_4$-set $S$ can be uniformly associated with an $s$-degree $b$ such that $S = S_b = \{i : a_i \leq_s b\}$.

Thus $S_b \subseteq S_c \Leftrightarrow L_s \models \forall a [\alpha(a, \overline{p}) \& a \leq b \rightarrow a \leq c]$.

Hence the first order theory of the poset $\langle \Sigma^0_4\text{-sets}, \subseteq \rangle$ (which is known to be hereditarily undecidable, see [9]) is elementarily definable with parameters in the $\Sigma^0_2$ s-degrees, giving undecidability of the local structure $L_s$, as stated in Corollary 2.4.

Next theorem shows the existence of an independent set of $\Sigma^0_2$ s-degrees which is definable with parameters.

**Theorem 2.2.** There is an independent set of $\Sigma^0_2$ s-degrees that is first-order definable with parameters. More specifically, there exist 2-c.e. $s$-degrees $\{g_i\}_{i \in \omega}$, g, a, b, such that the $g_i$'s form an independent set and are the minimal solutions of the inequalities

$$x \leq_s g \& a \leq_s x \cup b,$$

i.e. for every $i$, $a \leq_s g_i \cup b$, and for every $x$,

$$x \leq_s g \& a \leq_s x \cup b \Rightarrow (\exists i)[g_i \leq_s x].$$

Before proving this theorem, we state a few corollaries. We first recall the following theorem.

**Theorem 2.3** ([1]). Let $\mathfrak{P} = \langle P, \leq, \lor, 0 \rangle$ be an upper semilattice such that, for some $n \geq 1$, the partial order of $\Sigma^0_n$-sets under inclusion is first order definable with parameters in $\mathfrak{P}$. Then the first order theory of $\mathfrak{P}$ is undecidable.

It now follows that

**Corollary 2.4.** The first order theory of the $\Sigma^0_2$ s-degrees is undecidable.

**Proof.** This is clear from Theorem 2.3 and the discussion at the beginning of this section. \hfill \Box

Next, recall the following lemma:

**Lemma 2.5** ([18]). For every 2-c.e. set $C$ there exists a $\Pi^0_1$-set $D$ such that $C \equiv_s D$.

**Proof.** See [18]. \hfill \Box

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As a consequence we have

**Theorem 2.6.** There is an independent set of $\Pi^0_1$ s-degrees that is first-order definable with parameters. More specifically, there exist $\Pi^0_1$ s-degrees $\{g_i\}_{i \in \omega}$, $g$, $a$, $b$, such that the $g_i$'s form an independent set and are the minimal solutions of the equation

$$x \leq_s g \land a \leq_s x \cup b$$

i.e. for every $i$, $a \leq_s g_i \cup b$, and for every $\Pi^0_1$-degree $x$,

$$x \leq_s g \land a \leq_s x \cup b \Rightarrow (\exists i)[g_i \leq_s x].$$

**Proof.** By Theorem 2.2, and Lemma 2.5.

Lastly, we recall the following version of the Exact Degree Theorem.

**Theorem 2.7** ([13]). Let $A$ be a $\Pi^0_1$-set such that the $s$-degrees of its columns $\{A^{[i]}\}$ form an independent antichain. Then for every $\Sigma^0_4$-set $S$, there uniformly exists a $\Pi^0_1$-set $C$ such that, for every $i$,

$$i \in S \Leftrightarrow A_i \leq_s C.$$ 

**Proof.** Nies proves the Exact Degree Theorem for c.e. $Q$-degrees: Namely, he shows that starting from any c.e. set $G$ such that the $Q$-degrees of the columns of $G$ form an independent antichain in the $Q$-degrees, then for every $\Sigma^0_4$-set $S$, there one can uniformly find a c.e. set $C$ such that, for every $i$,

$$i \in S \Leftrightarrow A_i \leq_s C.$$ 

Then the result translates to an Exact Degree Theorem for $\Pi^0_1$ s-degrees by the isomorphism between c.e. $Q$-degrees and $\Pi^0_1$ s-degrees established by Lemma 1.2.

This gives us the last two corollaries.

**Corollary 2.8.** The $\Pi^0_1$ s-degrees are undecidable.

**Proof.** By Theorem 2.6, and Theorem 2.7.

Hence, we get as a corollary a different proof of a result due to Downey, LaForte and Nies, [19]:

**Corollary 2.9** ([19]). The first order of the c.e. $Q$-degrees is undecidable.

**Proof.** Again, by the isomorphism between c.e. $Q$-degrees and $\Pi^0_1$ s-degrees, established by Lemma 1.2.

### 3 A first order definable independent antichain

In this section, we prove Theorem 2.2 which gives us a first order definable independent set.

We aim at constructing 2-c.e. sets $A$, $B$, and $G_i$, with $i \in \omega$ such that the following requirements are satisfied, where $G = \bigoplus G_i$. 6
The requirements. The construction aims at satisfying the following requirements:

\[ D_i : (\exists \Delta_i)[A = \Delta_i^{G_i \oplus B_i}] \]
\[ I_{i,\Phi} : G_i \neq \Phi[G_i] \]
\[ M_{\Phi, \Psi} : A = \Phi^{G_i \oplus B} \Rightarrow (\exists i)(\exists \Gamma_i)[G_i = \Gamma_i^{\Phi G_i}] \]

where \( \Phi, \Psi \) are given \( s \)-operators, and \( \Delta_i, \Gamma_i \) are \( s \)-operators built by us, and \( G_i \neq \oplus G_i, G_j \). Then it is easy to show that the \( s \)-degrees \( g = \deg_s(G), g_i = \deg_s(G_i), a = \deg_s(A), \) and \( b = \deg_s(B) \) satisfy the claim.

Informal description of the strategies: Before giving the formal construction we give some intuition underlying the strategies used to meet the requirements.

The strategy for requirement \( D_i \). The strategy here consists in contributing to the definition of a correct \( s \)-operator \( \Delta_i \) such that \( A = \Delta_i^{G_i \oplus B_i} \). Imagine we have placed this strategy on a tree of strategies: Let us call this strategy \( \alpha \).

(For the sake of definiteness, we employ here terminology and notions concerning trees, which will be fully introduced later.) Then \( \alpha \) defines suitable axioms of the form \( \langle x, \emptyset \oplus \emptyset \rangle \in \Delta_i \) for all those numbers \( x \) that higher priority strategies (i.e., strategies \( \beta \leq \alpha \)) want to restrain in \( A \). When \( \alpha \) acts, it initializes all strategies \( \beta > \alpha \), thus assuming that any witness \( x \) used by any such \( \beta \) before its initialization will maintain its \( A \)-membership state forever (i.e., \( x \in A \) if and only if currently \( x \in A \)), and for any such witness \( x \in A, \alpha \) defines the axiom \( \langle x, \emptyset \oplus \emptyset \rangle \in \Delta_i \). Finally \( \alpha \) lets strategies \( \beta \supset \alpha \) maintain a correct definition of \( \Delta_i \) with respect to the elements that these \( \beta \)'s are using: More specifically, such a \( \beta \) may define an axiom of the form \( \langle x, \{ g \} \oplus \emptyset \rangle \in \Delta_i \), defining \( g \in G \), and then later possibly an axiom of the form \( \langle x, \emptyset \oplus \{ b \} \rangle \in \Delta_i \), defining \( b \in B \). If later \( \beta \) wants to extract \( x \) from \( A \), then \( \beta \) also needs to extract \( g \) from \( G \) and \( b \) from \( B \).

The strategy for requirement \( I_{i, \Phi} \). This is a more or less obvious version for \( s \)-reducibility of the classical Friedberg-Muchnick strategy:

1. Appoint a new witness \( g \in G_i \);

2. Await \( g \in \Phi[G_i] \). If and when this happens, through say an axiom \( \langle g, F \rangle \in \Phi \) with \( F \subseteq G_i \), then extract \( g \) from \( G_i \), and restrain \( F \subseteq G_i \): This is possible since \( g \notin F \).

The strategy for requirement \( M_{\Phi, \Psi} \). Suppose that \( \Phi \) and \( \Psi \) are given \( s \)-operators. At first we try to diagonalize, and to define \( A, G, \) and \( B \) in such a way as to have, for some \( x, A(x) \neq \Phi^{G_i \oplus B}(x) \). So the first attempt consists in trying to execute the following actions:

1. Appoint a witness \( x \), and temporarily let \( x \in A \);

\[ 1 \text{ Note that } \emptyset \oplus \emptyset \text{ is just a more informative way of denoting the empty set } \emptyset! \]

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2. await $x \in \Phi^{G \oplus B}$; (this will be referred to as outcome $w$.)

3. extract $x$ from $A$, and restrain $x \in \Phi^{G \oplus B}$. (This will be referred to as outcome $d$.)

Unfortunately, it might not be possible to proceed with item (3) of the previous naive strategy. Indeed, the following could happen, as a consequence of the interaction of our strategy with $D$-strategies having higher priority: Axioms of the form $\langle x, \{y\} \oplus \emptyset \rangle \in \Phi$, and $\langle y, \{g\} \rangle \in \Psi$ might appear, with $g \in G$, which makes $x \in \Phi^{G \oplus B}$, but on the other hand there is a higher priority strategy $D_i$, with an axiom $\langle x, \{g\} \oplus \emptyset \rangle \in \Delta_i$ already defined, thus with $g \in G_i$, so it is not possible to restrain $g \in G$ (which would make $x \in \Delta_i^{G_i \oplus B}$), and extract $x$ from $A$, without injuring $D_i$. So, unless later axioms of a different form appear for $y$ in $\Psi$ (for instance: $\langle y, \emptyset \rangle \in \Psi$; or $\langle y, \{g'\} \rangle \in \Psi$, with $g' \neq g$ such that $g'$ can be restrained without preventing $D_i$ from extracting $x$; or later axioms of the form $\langle x, \emptyset \oplus \emptyset \rangle \in \Phi$, or $\langle x, \emptyset \oplus \{b\} \rangle \in \Phi$ with $\langle x, \emptyset \oplus \{b\} \rangle \notin \Delta_i$), we do the following:

We define an $s$-operator $\Gamma_i$, by enumerating the axiom $\langle y, \{y\} \rangle \in \Gamma_i$; and extract from $G$ all those numbers $\hat{g}$ such that there are axioms $\langle x, \{\hat{g}\} \oplus \emptyset \rangle \in \Delta_j$ for all strategies $D_j$, with $j \neq i$ of higher priority than $M_{B, \Psi}$. If a new axiom $\langle y, \{g'\} \rangle \in \Psi$ (with $g' \in G$) as before appears, then $g'$ is different from $g$ and the $\hat{g}$'s, and we are free to diagonalize as explained above, by restraining $g' \in G$, extracting $x$ from $A$, and correcting all $\Delta_i$'s. On the other hand, if no new such axiom appears then we have $g \in G_i$ if and only if $y \in G^G_i$, and thus $g \in G_i$, if and only if $g \in \Gamma_i^G$. The idea is then to “pass on” $g$ (through a sort of stream of elements) to lower priority strategies for their own use. Whatever they do with $g$, they can not destroy correctness of $\Gamma_i$ at $g$. Unfortunately, if no further action is taken this would make $x \notin \Delta_j^{G_j \oplus B}$ for $j \neq i$. To set $A(x) = \Delta_j^{G_j \oplus B}(x)$ for all relevant $j$, we select a new element $b$, define $b \in B$, and enumerate the axiom $\langle x, \emptyset \oplus \{b\} \rangle \in \Delta_j$. Of course if later we are able to diagonalize by extracting $x$ from $A$, then we must extract $b$ from $B$, together with $g$, in order to preserve $A(x) = \Delta_j^{G_j \oplus B}(x)$.

Having lost $x$ as a diagonalization witness, we then appoint a new witness $x'$ in a new attempt at diagonalization as before. Proceeding as outlined above, if all our attempts at diagonalization fail, then since there are only finitely many strategies $D_j$ having higher priority than $M_{B, \Psi}$, the conclusion must be that there is a least $i$ such that we define infinitely many axioms of the form $\langle g, \{y\} \rangle \in \Gamma_i$, and the infinite set (stream) of these $g$'s can be used as witnesses by lower priority strategies. There are of course other stratagems that one has to employ here. In particular, $\Gamma_i$ is a priori correct only on the $g$'s that are in the stream. We must make sure that $\Gamma_i$ is correct also on numbers which are not in the stream. This is no problem as regards numbers used as witnesses by higher priority strategies. On the other hand when we define $\Gamma_i$ we initialize all strategies of lower priority that may use numbers not in the stream, and our strategy assumes that these numbers will maintain their $G_i$-membership, thus defining an axiom $\langle g', \emptyset \rangle \in \Gamma_i$ for those relevant $g'$'s that are currently in $G_i$. 

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The tree of strategies. We work with a tree of strategies

\[ T \subseteq (\omega \cup \{d, w\})^{<\omega} \]

where the set of outcomes (i.e. the elements of \(\omega \cup \{w, d\}\)) are ordered as follows:

\[ d < 0 < 1 < \cdots < w. \]

We will refer to some computable requirement assignment \(R\) of requirements to the elements of \(T\) (i.e. finite strings of outcomes; the strings in \(T\) are also called nodes, or strategies), i.e. a function \(R\) mapping nodes to requirements (we will denote by \(R_\alpha\) the requirement assigned to node \(\alpha\), in such a way that along any infinite path of \(T\), \(R\) is in fact a bijection. We say that a strategy \(\alpha\) is a \(D\)-strategy (\(I\)-strategy, \(M\)-strategy, respectively) if \(R_\alpha = D_i\) for some \(i\) (\(R_\alpha = I_i,\Phi\), \(R_\alpha = M_{i,\Phi}\), respectively, for some \(i\) and some pair of \(s\)-operators \(\Phi, \Psi\)). We assume that if \(\beta \subset \beta'\) are \(D\)-strategies and \(D_\beta = D_i, D_\beta' = D_{i'}\) then \(i < i'\). If \(\beta\) is a strategy such that, e.g., \(R_\beta = D_i\), then we also write \(G_\beta\) for \(G_i\); and similar other abuses of notations will be allowed, hopefully without affecting clearness and readability of the proof.

We use the standard notations and terminology on strings. In particular, given strings \(\alpha, \beta \in T\): \(|\alpha|\) denotes the length of \(\alpha\); \(\alpha \subseteq \beta\) means that \(\alpha\) is an initial segment of \(\beta\); \(\alpha \prec_L \beta\) means that there is a string \(\gamma \subset \alpha, \beta\) and \(\alpha(|\gamma|) < \beta(|\gamma|)\); \(\alpha \preceq \beta\) means \(\alpha \subseteq \beta\) or \(\alpha \prec_L \beta\), etc.; we say that \(\alpha\) has higher priority than \(\beta\) if \(\alpha < \beta\); the empty string is denoted by \(\lambda\).

During the construction, we define approximations to the sets \(G_i, A, B\). We will guarantee that \(G_i \subseteq \omega[\omega]\), so that in fact \(G = \bigoplus_{i \in \omega} G_i\) can be taken to be the union of the \(G_i\)'s. We define also several auxiliary sets, \(s\)-operators, parameters, etc. In particular:

- for every \(D\)-node \(\alpha\) we define an \(s\)-operator \(\Delta_\alpha\); for every \(M\)-node \(\alpha\) and any \(i\), an \(s\)-operator \(\Gamma_{\alpha,i}\);
- for every \(I\)-node we define a witness \(g_\alpha\); for every \(M\)-node \(\alpha\) we define witnesses \(x_\alpha(0), x_\alpha(1), \ldots\), and parameters \(b_\alpha(0), b_\alpha(1), \ldots\);
- for every \(\alpha\) we define a set (called stream) \(S_\alpha\), which is given, stage by stage, by specifying its elements.

At stage \(s\) of the construction, in addition to the approximations \(G_{i,s}, A_s, B_s\) to the sets \(G_i, A, B\), respectively, we define approximations to the above mentioned parameters, thus defining \(\Delta_{\alpha,s}, \Gamma_{\alpha,i,s}, g_{\alpha,s}, x_{\alpha,s}(t), b_{\alpha,s}(t), S_{\alpha,s}\), etc.

The desired sets \(G_i, A, B,\) and \(G\) will eventually be defined by

\[ G_i = \{ y : (\exists t)(\forall s \geq t)[y \in G_{i,s}] \}, \]
\[ A = \{ y : (\exists t)(\forall s \geq t)[y \in A_s] \}, \]
\[ B = \{ y : (\exists t)(\forall s \geq t)[y \in B_s] \}, \]

and, as already remarked, \(G = \bigcup_{i \in \omega} G_i\).
Definition 3.1. When we initialize a strategy $\alpha$ at stage $s$, we discard the current version of the relative parameters i.e. we set $\Delta_{\alpha,s} = \Gamma_{\alpha,i,s} = S_\alpha(s) = \emptyset,$ $g_{\alpha,s} = \uparrow$ (undefined), and $x_{\alpha,s}(t) = b_{\alpha,s}(t) = \uparrow$ for any $t$. (Hence when we initialize $\alpha$ we discard the current values of the parameters, waiting to define new values if needed later. Notice that upon discarding the value of a parameter, the construction will not change its current membership state, i.e. $x_{\alpha,s}(t) \in A$ if currently in $A$, or it will be forever $x_{\alpha,s}(t) \notin A$ if currently not in $A$, etc.)

In order to define $S_{\alpha,s}$, we will in fact define $S_{\alpha,s}^{[j]}$, i.e. $S_{\alpha,s} \cap \omega^{[j]}$, for every $j$:

The idea underlining the set $S_{\alpha}^{[j]}$ is that the only elements that strategies $\beta \supseteq \alpha$ may use in order to define $G_j$ are taken from $S_{\alpha}^{[j]}$.

Definition 3.2. During the construction we say that at a stage $s+1$ a number $y$ is new for strategy $\alpha$ if either

1. $y$ needs to be chosen to be enumerated into $A$ or $B$, and $y$ is bigger than any number that has been used so far by any other strategy; or
2. $y$ needs to be chosen to be enumerated into $G_i$, for some $i$, and $y \in S_{\alpha}^{[i]}[s+1] - S_{\alpha}^{[i]}[s]$.

In the construction below, any parameter different from $S_{\lambda,s}$ retains the same value as at the previous stage unless otherwise specified. And unless explicitly stated otherwise, at any stage $S_{\lambda,s} = S_{\lambda}$, for any outcome $o$.

The construction. By stages: We define at stage $s$ a string $\delta_s$, which is the current approximation to what will be called the true path.

Step 0. Let $\delta_0 = \lambda$. Initialize all strategies.

Step $s+1$. For the sake of simplicity we will often write $p$ (where $p$ is a parameter) instead of $p(s)$ or $p(s+1)$ to denote the most recent value of $p$ that has been defined, or is being defined, during the construction. We also often omit the strategy to which the parameter refers, thus writing $p$ for $p_{\alpha}$, when the strategy is clearly understood from the context. Similarly, we omit specifying $s$ when writing $x \in A$, meaning $x \in A_s$, etc.

Suppose we have already defined $\alpha = \delta_{s+1} \upharpoonright n$, and $S_{\alpha}^{[j]}$, for every $j$, having defined $S_{\lambda}[s+1] = [0,s]$.

We act on $\alpha$ according to the requirement $R_{\alpha}$.

$R_{\alpha} = D_i$: For any $x \in A$ enumerated into $A$ by any $\beta \not\supseteq \alpha$, add the axiom $\langle x, \emptyset \oplus \emptyset \rangle \in \Delta_{\alpha}$. Let $\alpha^{-}\langle 0 \rangle$ be eligible to act next.

$R_{\alpha} = I_{i,\emptyset}$: We distinguish the following cases:

1. There is no appointed witness: If there is a new $g$ for $\alpha$ then appoint the least such $g$ as witness, define $g \in G_i$, and let $\alpha^{-}\langle w \rangle$ be eligible to act next. Otherwise, end the stage.

2. $g \in G_i = \Phi^{G_x}$: Let $\alpha^{-}\langle w \rangle$ be eligible to act next.
3. \( g \in \Phi^{G_{x_{i}}} \): Define \( g \notin G_{i} \). If this is the first time we have taken this case since the last initialization of \( \alpha \), end the stage. (This has the effect of restraining \( g \in \Phi^{G_{x_{i}}} \) if \( \alpha \) is never again initialized.) Otherwise, let \( \alpha^{-}(d) \) be eligible to act next.

\( R_{\alpha} = M_{\Phi, \Psi} \): We first give the following definition which allows us to identify an \( x \) that we can force \( x \in \Phi^{G_{\oplus B} - A} \), and still make \( A(x) = \Delta^{G_{B} \oplus B}_{\beta}(x) \) for each \( D \)-strategy \( \beta \subset \alpha \).

**Definition 3.3.** We say that a number \( x \) is **eligible to act at \( \alpha \)** if one of the following holds:

1. There is an axiom \( \langle x, \emptyset \oplus \emptyset \rangle \in \Phi \).
2. There is an axiom \( \langle x, \emptyset \oplus \{b\} \rangle \in \Phi \) such that \( \langle x, \emptyset \oplus \{b\} \rangle \notin \Delta_{\beta} \), for any \( \beta \subset \alpha \).
3. There is an axiom \( \langle x, \{y\} \oplus \emptyset \rangle \in \Phi \) and an axiom \( \langle y, \emptyset \rangle \in \Psi \).
4. There is an axiom \( \langle x, \{y\} \oplus \emptyset \rangle \in \Phi \) and an axiom \( \langle y, \{g\} \rangle \in \Psi \) with \( g \in G \), such that there is no \( D \)-node \( \beta \subset \alpha \), with the axiom \( \langle x, \{y\} \oplus \emptyset \rangle \in \Delta_{\beta} \).

We now proceed with the strategy. Suppose that since last initialization of \( \alpha \) we have already defined \( x_{\alpha}(t) \) and \( b_{\alpha}(t) \), with \( t < n \). We distinguish the following cases:

1. If some \( x = x_{\alpha}(t) \) is eligible to act, do the following **actions**: 
   Extract \( x \) from \( A \), i.e. define \( x \notin A \). 
   Correct \( \Delta_{\beta} \) for \( \beta \subset \alpha \): If \( \langle x, \{y\} \oplus \emptyset \rangle \in \Delta_{\beta} \) then extract \( y \) from \( G_{\beta} \), if \( \langle x, \emptyset \oplus \{b\} \rangle \in \Delta_{\beta} \) then extract \( b \) from \( B \). If this is the first time we have taken this outcome since \( \alpha \)'s last initialization, end the current stage. Otherwise, let \( \alpha^{-}(d) \) be eligible to act next.

2. \( n > 0 \) and \( x_{\alpha}(n - 1) \in A - \Phi^{G_{\oplus B}} \): let \( \alpha^{-}(w) \) be eligible to act next.

3. Otherwise \( x_{\alpha}(n - 1) \in \Phi^{G_{\oplus B} \cap A} \) (denote again for simplicity \( x = x_{\alpha}(n - 1) \)). We further distinguish two cases:

   a) \( b_{\alpha}(n - 1) \) is undefined. Notice that there are only axioms of the form \( \langle x, \{y\} \oplus \emptyset \rangle \in \Phi \), such that for all axioms \( \langle g, \{g\} \rangle \in \Psi \) there is a \( D \)-node \( \beta \subset \alpha \), with \( g \in G_{\beta} \), and an axiom \( \langle x, \{y\} \oplus \emptyset \rangle \in \Delta_{\beta} \), so that we can not restrain \( g \in G_{\beta} \), and extract \( x \) from \( A \) without making it impossible to achieve \( A(x) = \Delta^{G_{\oplus B}}_{\beta} \). Pick the least such \( \beta \), and suppose that \( R_{\beta} = D_{i} \): For all \( D \)-strategies \( \beta' \neq \beta \) such that \( \beta' \subset \alpha \), define \( g' \notin G_{\beta', x+1} \), where \( g' \) is such that there is the axiom \( \langle x, \{g'\} \oplus \emptyset \rangle \in \Delta_{\beta'} \). Pick a new \( b = b_{\alpha}(n - 1) \), define \( b \in B \), and add the axiom \( \langle x, \emptyset \oplus \{b\} \rangle \in \Delta_{\beta'}, \) for any \( D \)-node \( \beta' \subset \alpha \).
   Add the axiom \( \langle g, \{y\} \rangle \in \Gamma_{\alpha,i} \). For each \( g' \) such that \( g' \in G_{i} \) and \( g' \) has been enumerator by some strategy \( \beta \notin \alpha^{-}(i) \), define the axiom
A careful look at the construction shows that if $\alpha\prec(i)$ is any of the sets $G_i$, $A$, $B$ are 2-c.e.

**Proof.** A careful look at the construction shows that if $X$ is any of the sets $G_i$, $A$, $B$, then for every $x$, at stage 0 we have $X_0(x) = 0$.

Consider first the case $X = A$. An element $x$ can enter $A$ only if enumerated by some $M$-strategy $\alpha$, i.e. $x = x_{\alpha}(t)$, for some $t$. But then it can only be extracted just through Case 1 of the same strategy $\alpha$. After this $x$ is never again enumerated into $A$.

A similar argument applies to $B$. An element $b$ can enter $B$ if enumerated by some $M$-strategy $\alpha$, and can be extracted again only by $\alpha$ upon giving outcome $d$.

Finally assume that $X = G_i$. An element $g$ can be enumerated in $G_i$ the first time by an $M$-strategy $\alpha$, in correspondence to some witness $x$, i.e. $\alpha$ enumerates $x$ into $A$, defines the axiom $\langle x, \{g\} \oplus \emptyset \rangle \in \Delta_\alpha$ and defines $g \in G_i$. Then it can only be extracted by the same strategy $\alpha$, when moving to outcome $d$ or to outcome $j$ with $j \neq i$; or it can be extracted by some strategy $\gamma \supseteq \alpha\prec(i)$, after $g$ has been put in the set $S_{\alpha\prec(i)}$. After being extracted $g$ is not used anymore.

**Lemma 4.2.** For every $n$ the following hold: $\alpha_n = \liminf_\delta \delta_n \mid \exists n \exists \alpha_n$ is eventually never initialized; for every $j$, after the last initialization of $\alpha_n$ there are infinitely many $\alpha_n$-true stages $s$ (i.e. stages at which $\alpha_n \subseteq \delta_n$) such that $S_{\alpha_n,s}$ contains a new element; witnesses $g_{\alpha_n}$, and $x_{\alpha_n}(t), b_{\alpha_n}(t)$ reach a limit.

**Proof.** The proof is by induction on $n$. For $n = 0$ the claim is obvious.

Suppose now that $\alpha_n = \liminf_\delta \delta_n \mid \exists n \exists \alpha_n$ is eventually never initialized, and the inductive claim is true of $n$. For simplicity, let $\alpha = \alpha_n$. Let $t$ be a stage such that at no $s \geq t$ do we act on any $\beta <_L \alpha$. Notice that the inductive assumption on $S_{\alpha_n}^{[j]}$ allows us to conclude that if $\alpha$ needs to appoint some new element $g \in S_{\alpha_n}^{[j]}$ in order to define the set $G_j$, then eventually it is allowed to do so.

We now distinguish three cases according to whether $\alpha$ is a $D$-strategy, or an $I$-strategy, or an $M$-strategy, respectively.

\[\langle g', \emptyset \rangle \in \Gamma_{\alpha,i}\] Define $S_{\alpha\prec(i)}^{[j]}[s + 1] = S_{\alpha\prec(i)}^{[j]}[s] \cup \{g\}$ (notice that $S_{\alpha\prec(i)}^{[j]}[s + 1] = S_{\alpha\prec(i)}^{[j]}[s]$ if $j \neq i$) and let $\alpha\prec(i)$ be eligible to act next.

(b) $b = b_n(n - 1)$ is defined: Choose a new $x_{\alpha}(n)$ and define $x_{\alpha}(n) \in A$.

i. If for every $D$-strategy $\beta \subset \alpha$, with say $D_\beta = D_j$, there is a new number $g_j \in S_{\alpha}^{[j]}$, then define $g_j \in G_j$ and add the axiom $\langle x, \{g_j\} \oplus \emptyset \rangle \in \Delta_\beta$. End the current stage.

ii. Otherwise (while waiting for enough new numbers), end the stage.

4 Verification of the construction

**Lemma 4.1.** The sets $G_i$, $A$ and $B$ are 2-c.e.
Lemma 4.2: We first notice that when $\alpha$ acts, we give outcome 0, and we never end the stage after acting. So

$$\alpha_{n+1} = \liminf_s \delta_s \upharpoonright n + 1 = \alpha^\sim(0).$$

On the other hand we always have $S^{[j]}_{\alpha_{n+1}} = S^{[j]}_\alpha$, any $j$.

$R_\alpha = D_i$: Let $x$ be given. In order to check that $A(x) = \Delta^G_{\alpha} \oplus B(x)$, we need only check this for those numbers $x$ such that there are a $\beta$ and $t$ with $x = x_\beta(t)$. Only strategy $\beta$ is responsible for keeping $x$ in or out of $A$. Without loss of generality, we may assume that $t \geq t_n$.

Case 1): $\beta \supsetneq \alpha$. At the first $\alpha$-stage $s > t$ if $x \in A$ then we enumerate the axiom $\langle x, \emptyset \oplus \emptyset \rangle \in \Delta_\alpha$, which makes $x \in \Delta^G_{\alpha} \oplus B$. Otherwise, if $x \notin A$ then at no $\alpha$-stage $s$ after last initialization of $\alpha$ do we have $x \in A$, hence we do not define any $\Delta_\alpha$-axiom for $x$.

Case 2): $\beta \supseteq \alpha$. At stage $t$, when $\beta$ appoints $x$, $\beta$ enumerates also an axiom $\langle x, \emptyset \oplus \emptyset \rangle \in \Delta_\alpha$, letting $g \in G_i$, which makes $x \in \Delta^G_{\alpha} \oplus B$ as long as $\beta$ takes outcome $w$, waiting for $x \in \Phi^G_{\beta} \oplus B$. Then either $\beta$ takes immediately outcome $d$, extracts $x$ from $A$ and $g$ from $G_i$, which makes $A(x) = \Delta^G_{\alpha} \oplus B(x)$; or $\beta$ takes some outcome $j \in \omega$, keeps $x \in A$, adds an axiom $\langle x, \emptyset \oplus \{b\} \rangle$ in $\Delta_\alpha$, which keeps $x \in \Delta^G_{\alpha} \oplus B$ even if some lower priority strategy extracts $g$ from $G_i$ until $\beta$ may take outcome $d$ and thus extract $x$ from $A$, $g$ from $G_i$ and $b$ from $B$, making $x \notin \Delta^G_{\alpha} \oplus B$. In all cases $A(x) = \Delta^G_{\alpha} \oplus B(x)$.

$R_\alpha = I_{i, \emptyset}$: Let $t$ be a stage after which $\alpha$ does not change $g_\alpha$, anymore. By Lemma 4.2 such a stage exists. If at no future $\alpha$-stage do we have $g_\alpha \in \Phi^{G_{x_t}}$.

Lemma 4.3. For every $n$, $R_{\alpha_n}$ is satisfied.

Proof. Let $\alpha = \alpha_n$, for some $n$, be given, and suppose by the previous lemma that $t_n$ is the last stage at which $\alpha$ is initialized.

$R_\alpha = D_i$: Let $x$ be given. In order to check that $A(x) = \Delta^G_{\alpha} \oplus B(x)$, we need only check this for those numbers $x$ such that there are a $\beta$ and $t$ with $x = x_\beta(t)$. Only strategy $\beta$ is responsible for keeping $x$ in or out of $A$. Without loss of generality, we may assume that $t \geq t_n$.

Case 1): $\beta \supsetneq \alpha$. At the first $\alpha$-stage $s > t$ if $x \in A$ then we enumerate the axiom $\langle x, \emptyset \oplus \emptyset \rangle \in \Delta_\alpha$, which makes $x \in \Delta^G_{\alpha} \oplus B$. Otherwise, if $x \notin A$ then at no $\alpha$-stage $s$ after last initialization of $\alpha$ do we have $x \in A$, hence we do not define any $\Delta_\alpha$-axiom for $x$.

Case 2): $\beta \supseteq \alpha$. At stage $t$, when $\beta$ appoints $x$, $\beta$ enumerates also an axiom $\langle x, \emptyset \oplus \emptyset \rangle \in \Delta_\alpha$, letting $g \in G_i$, which makes $x \in \Delta^G_{\alpha} \oplus B$ as long as $\beta$ takes outcome $w$, waiting for $x \in \Phi^G_{\beta} \oplus B$. Then either $\beta$ takes immediately outcome $d$, extracts $x$ from $A$ and $g$ from $G_i$, which makes $A(x) = \Delta^G_{\alpha} \oplus B(x)$; or $\beta$ takes some outcome $j \in \omega$, keeps $x \in A$, adds an axiom $\langle x, \emptyset \oplus \{b\} \rangle$ in $\Delta_\alpha$, which keeps $x \in \Delta^G_{\alpha} \oplus B$ even if some lower priority strategy extracts $g$ from $G_i$ until $\beta$ may take outcome $d$ and thus extract $x$ from $A$, $g$ from $G_i$ and $b$ from $B$, making $x \notin \Delta^G_{\alpha} \oplus B$. In all cases $A(x) = \Delta^G_{\alpha} \oplus B(x)$.

$R_\alpha = I_{i, \emptyset}$: Let $t$ be a stage after which $\alpha$ does not change $g_\alpha$, anymore. By Lemma 4.2 such a stage exists. If at no future $\alpha$-stage do we have $g_\alpha \in \Phi^{G_{x_t}}$. 

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then \( \alpha_n + 1 = \alpha \setminus \langle w \rangle \) and the requirement is satisfied. Otherwise at some future \( \alpha \)-stage we have that \( g_\alpha \in \Phi^{G_{x_i}} \). As explained in the construction, at the first such stage, we restrain \( g_\alpha \in \Phi^{G_{x_i}} \), and we extract \( g_\alpha \) from \( A \), thus letting \( g_\alpha \in \Phi^{G_{x_i}} - A \).

\[ R_\alpha = M_{\Phi, \Psi} : \text{If } \alpha_{n+1} = \alpha \setminus \langle w \rangle \text{ then there exists } n \text{ such that } x = x_\alpha(n) \text{ is defined, no } x_\alpha(m) \text{ is ever defined for } m > n \text{ and } x \in A - \Phi^{G_{x_i} \oplus B}. \text{ If } \alpha_{n+1} = \alpha \setminus \langle d \rangle \text{ then there is some } x = x_\alpha(t) \text{ (among finitely many witnesses } x_\alpha(0), \ldots, x_\alpha(n) \text{ which } \beta \text{ has defined after last initialization) such that } x \in \Phi^{G_{x_i} \oplus B} - A. \]

It remains to consider the case when \( \alpha_n + 1 = \alpha \setminus \langle i \rangle \text{ for some } i \in \omega \). We claim in this case that \( G_i = \Gamma^G_{\alpha,i} \), where \( =^* \) denotes equality modulo a finite set, and \( \Gamma_{\alpha,i} \) is the \( s \)-operator, as enumerated by \( \alpha \) after the last initialization of \( \alpha \).

If \( g \) is eventually used by a strategy \( \beta \leq \alpha \), then either \( g \notin G \), and in this case there is no axiom \( \langle g, F \rangle \in \Gamma_{\alpha,i} \), or \( g \in G \), in which case by construction we add an axiom \( \langle g, \emptyset \rangle \in \Gamma_{\alpha,i} \).

Next, for every \( g \) which is ever used by any strategy \( \beta > \beta \), \( \alpha \setminus \langle i \rangle \), we have (at the moment when we discard \( g \) by initialization) either \( g \notin G_i \), in which case we have \( G_i(g) = \Gamma^G_{\alpha,i}(g) \) since we never define any axiom in \( \Gamma_{\alpha,i} \) for these \( g \)'s, or \( g \in G_i \), in which case we add an axiom \( \langle g, \emptyset \rangle \in \Gamma_{\alpha,i} \).

So we need only show that for every \( g \) such that \( g \) is enumerated into \( S_{\alpha,i}^{[i]} \) at some \( \alpha \setminus \langle i \rangle \)-stage,

\[ g \in G_i \Leftrightarrow g \in \Gamma^G_{\alpha,i}. \]

The reason we enumerated \( g \) into \( S_{\alpha,i}^{[i]} \) at some stage \( t' \geq t_n \) is that we found an axiom \( \langle y, \{ g \} \rangle \in \Psi \), with \( g \in G_i \), in correspondence with some witness \( x \), for which there is an axiom \( \langle x, \{ g \} \oplus \emptyset \rangle \in \Delta_\beta \) (where \( \beta \subset \alpha \) is such that \( R_\beta = D_j \)). Moreover there is no other axiom \( \langle y, \{ g' \} \rangle \in \Psi \) with \( g' \in G \).

Indeed, such an axiom can not appear after \( t' \) since in this case we would be able to diagonalize and give outcome \( d \). If it is present at stage \( t' \), then since we give outcome \( i \) there must be an axiom \( \langle x, \{ g' \} \oplus \emptyset \rangle \in \Delta_\beta \) with \( j > i \) (here \( D_j = D_j' \), some \( \beta' \) such that \( \beta \subset \beta' \subset \alpha \)), but in this case we extract \( g' \) from \( G \) by construction.

We are now able to conclude:

\[ g \in G_i \Leftrightarrow y \in \Psi^G \Leftrightarrow g \in \Gamma^G_{\alpha,i} \]

as desired.

\[ \blacksquare \]

**Lemma 4.4.** Let \( g_i = \deg_s(G_i) \). The set \( \{ g_i : i \in \omega \} \) is first order definable with parameters \( a, b, c \).

**Proof.** Let \( \alpha(x) \) be the following formula with parameters \( g, a, b \) in the language of partial orders (where \( \lor \) and \( \prec \) are obvious abbreviations):

\[ \alpha(x) : \quad x \leq g \& a \leq x \lor b \]

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and let
\[ \phi(x) : \alpha(x) \land \neg(\exists w < x)\alpha(w). \]

Now, if the parameters \( g, a, b \) are interpreted with the \( s \)-degrees \( g = \text{deg}_s(G), a = \text{deg}_s(A), b = \text{deg}_s(B) \), respectively, then in the structure \( L_s \) of the \( \Sigma_2^0 \) \( s \)-degrees, we have that
\[ L_s \models \phi(x, g, a, b) \iff x \in \{ g_i : i \in \omega \}. \]

Indeed, the formula is certainly satisfied when \( x = g_i \), any \( i \), since each \( g_i \) is incomparable with all the others. On the other hand, if \( x \leq_s g \) and \( a \leq_s x \lor b \) and there is no \( y <_s x \) such that \( y \leq_s g \) and \( a \leq_s y \lor b \), then since \( g_i \leq_s x \) for some \( i \), we have that \( g_i = x \).

\[ \Box \]

References


